

# Inequalities for Euler-Mascheroni constant

Hongmin Xu and Xu You

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## Abstract

The aim of this paper is to establish new inequalities for the Euler-Mascheroni by the continued fraction method.

## 1 Introduction

Euler's Constant was first introduced by Leonhard Euler (1707-1783) in 1734 as the limit of the sequence

$$(1.1) \quad \gamma(n) := \sum_{m=1}^n \frac{1}{m} - \ln n.$$

It is also known as the Euler-Mascheroni constant. There are many famous unsolved problems about the nature of this constant (See e.g. the survey papers or books of R.P. Brent and P. Zimmermann[1], Dence and Dence[3], Havil[5] and Lagarias[8]). For example, it is a long-standing open problem if it is a rational number. A good part of its mystery comes from the fact that the known algorithms converging to  $\gamma$  are not very fast, at least, when they are compared to similar algorithms for  $\pi$  and  $e$ .

The sequence  $(r(n))_{n \in \mathbb{N}}$  converges very slowly toward  $\gamma$ , like  $(2n)^{-1}$ . Up to now, many authors are preoccupied to improve its rate of convergence, see e.g. [2, 3, 4, 6, 7, 9, 10, 11, 14] and references therein. We list some main results as follows:

$$\sum_{m=1}^n \frac{1}{m} - \ln \left( n + \frac{1}{2} \right) = \gamma + O(n^{-2}), \quad (\text{DeTemple, [4]})$$

$$\sum_{m=1}^n \frac{1}{m} - \ln \frac{n^3 + \frac{3}{2}n^2 + \frac{227}{240} + \frac{107}{480}}{n^2 + n + \frac{97}{240}} = \gamma + O(n^{-6}), \quad (\text{Mortici, [11]})$$

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$$\sum_{m=1}^n \frac{1}{m} - \ln \left( 1 + \frac{1}{2n} + \frac{1}{24n^2} - \frac{1}{48n^3} + \frac{23}{5760n^4} \right) = \gamma + O(n^{-5}), \quad (\text{Chen and Mortici, [2]})$$

Recently, Mortici and Chen[14] provided a very interesting sequence

$$\begin{aligned} \nu(n) = & \sum_{m=1}^n \frac{1}{m} - \frac{1}{2} \ln \left( n^2 + n + \frac{1}{3} \right) \\ & - \left( \frac{-\frac{1}{180}}{(n^2 + n + \frac{1}{3})^2} + \frac{\frac{8}{2835}}{(n^2 + n + \frac{1}{3})^3} + \frac{\frac{5}{1512}}{(n^2 + n + \frac{1}{3})^4} + \frac{\frac{592}{93555}}{(n^2 + n + \frac{1}{3})^5} \right), \end{aligned}$$

and proved

$$(1.2) \quad \lim_{n \rightarrow \infty} n^{12} (\nu(n) - \gamma) = -\frac{796801}{43783740}.$$

Hence the rate of the convergence of the sequence  $(\nu(n))_{n \in \mathbb{N}}$  is  $n^{-12}$ .

Very recently, by inserting the continued fraction term in (1.1), Lu[9] introduced a class of sequences  $(r_k(n))_{n \in \mathbb{N}}$  (see Theorem 1 below), and showed

$$(1.3) \quad \frac{1}{72(n+1)^3} < \gamma - r_2(n) < \frac{1}{72n^3},$$

$$(1.4) \quad \frac{1}{120(n+1)^4} < r_3(n) - \gamma < \frac{1}{120(n-1)^4}.$$

In fact, Lu[9] also found  $a_4$  without proof. In general, the continued fraction method could provide a better approximation than others, and has less numerical computations.

First, we will prove

**Theorem 1.** *For Euler-Mascheroni constant, we have the following convergent sequence*

$$r(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n - \frac{a_1}{n + \frac{a_2 n}{n + \frac{a_3 n}{n + \frac{a_4 n}{\ddots}}}},$$

where  $(a_1, a_2, a_4, a_6, a_8, a_{10}, a_{12}) = (\frac{1}{2}, \frac{1}{6}, \frac{3}{5}, \frac{79}{126}, \frac{7230}{6241}, \frac{4146631}{3833346}, \frac{306232774533}{179081182865})$ , and  $a_{2k+1} = -a_{2k}$  for  $1 \leq k \leq 6$ .

Let

$$R_k(n) := \frac{a_1}{n + \frac{a_2 n}{n + \frac{a_3 n}{n + \frac{a_4 n}{\ddots}}}},$$

(See Appendix for their simple expressions) and

$$r_k(n) := \sum_{m=1}^n \frac{1}{m} - \ln n - R_k(n).$$

For  $1 \leq k \leq 13$ , we have

$$(1.5) \quad \lim_{n \rightarrow \infty} n^{k+1} (r_k(n) - \gamma) = C_k,$$

where  $(C_1, \dots, C_{13}) = \left( -\frac{1}{12}, -\frac{1}{72}, \frac{1}{120}, \frac{1}{200}, -\frac{79}{25200}, -\frac{6241}{3175200}, \frac{241}{105840}, \frac{58081}{22018248}, -\frac{262445}{91974960}, \right.$   
 $\left. -\frac{2755095121}{892586949408}, \frac{20169451}{3821257440}, \frac{406806753641401}{45071152103463200}, -\frac{71521421431}{5152068292800} \right).$

**Open problem** For every  $k \geq 1$ , we have  $a_{2k+1} = -a_{2k}$ .

The main aim of this paper is to improve (1.3) and (1.4). We establish the following more precise inequalities.

**Theorem 2.** Let  $r_{10}(n), r_{11}(n)$ ,  $C_{10}$  and  $C_{11}$  be defined in Theorem 1, then

$$(1.6) \quad C_{10} \frac{1}{(n+1)^{11}} < \gamma - r_{10}(n) < C_{10} \frac{1}{n^{11}},$$

$$(1.7) \quad C_{11} \frac{1}{(n+1)^{12}} < r_{11}(n) - \gamma < C_{11} \frac{1}{n^{12}}.$$

*Remark 1.* In fact, Theorem 2 implies that  $r_{10}(n)$  is a strictly increasing function of  $n$ , whereas  $r_{11}(n)$  is a strictly decreasing function of  $n$ . Certainly, it has the similar inequalities for  $r_k(n)$  ( $1 \leq k \leq 9$ ), we leave these for readers to verify. It is also should be noted that (1.4) cannot deduce the monotony of  $r_3(n)$ .

*Remark 2.* It is worth to pointing out that Theorem 2 provides sharp bounds for harmonic sequence, which are superior to Theorem 3 and 4 of Mortici and Chen[14].

## 2 The Proof of Theorem 1

The following lemma gives a method for measuring the rate of convergence, This Lemma was first used by Mortici[12, 13] for constructing asymptotic expansions, or to accelerate some convergences. For proof and other details, see, e.g., [13].

**Lemma 1.** If the sequence  $(x_n)_{n \in \mathbb{N}}$  is convergent to zero and there exists the limit

$$(2.1) \quad \lim_{n \rightarrow +\infty} n^s (x_n - x_{n+1}) = l \in [-\infty, +\infty]$$

with  $s > 1$ , then there exists the limit:

$$(2.2) \quad \lim_{n \rightarrow +\infty} n^{s-1} x_n = \frac{l}{s-1}.$$

In the sequel, we always assume  $n \geq 2$ .

We need to find the value  $a_1 \in \mathbb{R}$  which produces the most accurate approximation of the form

$$(2.3) \quad r_1(n) = \sum_{m=1}^n \frac{1}{m} - \ln n - \frac{a_1}{n},$$

here we note  $R_1(n) = \frac{a_1}{n}$ . To measure the accuracy of this approximation, we usually say that an approximation (2.3) is better as  $r_1(n) - \gamma$  faster converges to zero. Clearly

$$(2.4) \quad r_1(n) - r_1(n+1) = \ln \left( 1 + \frac{1}{n} \right) - \frac{1}{n+1} + \frac{a_1}{n+1} - \frac{a_1}{n}.$$

It is well-known that for  $|x| < 1$ ,

$$\ln(1+x) = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{x^m}{m} \quad \text{and} \quad \frac{1}{1-x} = \sum_{m=0}^{\infty} x^m.$$

Developing the expression (2.4) into power series expansion in  $\frac{1}{n}$ , we easily obtain

$$(2.5) \quad r_1(n) - r_1(n+1) = \left( \frac{1}{2} - a_1 \right) \frac{1}{n^2} + \left( a_1 - \frac{2}{3} \right) \frac{1}{n^3} + \left( \frac{3}{4} - a_1 \right) \frac{1}{n^4} + O \left( \frac{1}{n^5} \right).$$

From Lemma 1, we see that the rate of convergence of the sequence  $(r_1(n) - \gamma)_{n \in \mathbb{N}}$  is even higher as the value  $s$  satisfying (2.1). By lemma 1, we have

(i) If  $a_1 \neq \frac{1}{2}$ , then the rate of convergence of the  $(r_1(n) - \gamma)_{n \in \mathbb{N}}$  is  $n^{-1}$ , since

$$\lim_{n \rightarrow \infty} n (r_1(n) - \gamma) = \frac{1}{2} - a_1 \neq 0.$$

(ii) If  $a_1 = \frac{1}{2}$ , from (2.5) we have

$$r_1(n) - r_1(n+1) = -\frac{1}{6} \frac{1}{n^3} + O \left( \frac{1}{n^4} \right).$$

Hence the rate of convergence of the  $(r_1(n) - \gamma)_{n \in \mathbb{N}}$  is  $n^{-2}$ , since

$$\lim_{n \rightarrow \infty} n^2 (r_1(n) - \gamma) = -\frac{1}{12}.$$

We also observe that the fastest possible sequence  $(r_1(n))_{n \in \mathbb{N}}$  is obtained only for  $a_1 = \frac{1}{2}$ .

Just as Lu[9] did, we may repeat the above approach to determine  $a_1$  to  $a_4$  step by step. However, the computations become very difficult when  $k \geq 5$ . In this paper we will use the *Mathematica* software to manipulate symbolic computations.

Let

$$(2.6) \quad r_k(n) = \sum_{m=1}^n \frac{1}{m} - \ln n - R_k(n),$$

then

$$(2.7) \quad r_k(n) - r_k(n+1) = \ln \left( 1 + \frac{1}{n} \right) - \frac{1}{n+1} + R_k(n+1) - R_k(n).$$

It is easy to get the following power series

$$(2.8) \quad \ln \left( 1 + \frac{1}{n} \right) - \frac{1}{n+1} = \sum_{m=2}^{\infty} (-1)^m \frac{m-1}{m} \frac{1}{n^m}.$$

Hence the key step is to expand  $R_k(n+1) - R_k(n)$  into power series in  $\frac{1}{n}$ . Here we use some examples to explain our method.

**Step 1:** . For example, given  $a_1$  to  $a_7$ , find  $a_8$ . Define

$$(2.9) \quad R_8(n) = \frac{\frac{1}{2}}{n + \frac{\frac{n}{6}}{n + \frac{-\frac{n}{6}}{n + \frac{\frac{3}{5} * n}{n + \frac{-\frac{3}{5} * n}{n + \frac{\frac{79}{126} * n}{n + \frac{-\frac{79}{126} * n}{n + a_8}}}}}}}} = \frac{-237 + 1405a_8 + 1800n + 1740a_8n - 630n^2 + 3780a_8n^2 + 3780n^3}{6(79a_8 + 600a_8n + 600n^2 + 790a_8n^2 + 1260a_8n^3 + 1260n^4)}.$$

By using the *Mathematica* software(The *Mathematica Program* is very similar to one given in Remark 3 below, however it has a parameter  $a_8$ ), we obtain

$$(2.10) \quad \begin{aligned} & R_8(n+1) - R_8(n) \\ &= -\frac{1}{2n^2} + \frac{2}{3n^3} - \frac{3}{4n^4} + \frac{4}{5n^5} - \frac{5}{6n^6} + \frac{6}{7n^7} - \frac{7}{8n^8} \\ &+ \frac{360030 - 6241a_8}{396900n^9} + \frac{-346440 + 24964a_8 + 6241a_8^2}{352800n^{10}} + O\left(\frac{1}{n^{11}}\right). \end{aligned}$$

Substituting (2.8) and (2.10) into (2.7), we get

$$(2.11) \quad \begin{aligned} r_8(n) - r_8(n+1) &= \left( -\frac{8}{9} + \frac{360030 - 6241a_8}{396900} \right) \frac{1}{n^9} \\ &+ \left( \frac{9}{10} + \frac{-346440 + 24964a_8 + 6241a_8^2}{352800} \right) \frac{1}{n^{10}} + O\left(\frac{1}{n^{11}}\right). \end{aligned}$$

The fastest possible sequence  $(r_8(n))_{n \in \mathbb{N}}$  is obtained only for  $a_8 = \frac{7230}{6241}$ . At the same time, it follows from (2.11),

$$(2.12) \quad r_8(n) - r_8(n+1) = \frac{58081}{2446472} \frac{1}{n^{10}} + O\left(\frac{1}{n^{11}}\right),$$

the rate of convergence of the  $(r_8(n) - \gamma)_{n \in \mathbb{N}}$  is  $n^{-9}$ , since

$$\lim_{n \rightarrow \infty} n^9 (r_8(n) - \gamma) = -\frac{58081}{22018248}.$$

We can use the above approach to find  $a_k$  ( $3 \leq k \leq 8$ ). Unfortunately, it does not work well for  $a_9$ . Since  $a_3 = -a_2$ ,  $a_5 = -a_4$  and  $a_7 = -a_6$ . So we may conjecture  $a_9 = -a_8$ . Now let's check it carefully.

**Step 2:** Check  $a_9 = -\frac{7230}{6241}$  to  $a_{13} = -\frac{306232774533}{179081182865}$ .

Let  $a_1, \dots, a_9$ , and  $R_9(n)$  be defined in Theorem 1. Applying the *Mathematica* software, we obtain

$$(2.13) \quad \begin{aligned} & R_9(n+1) - R_9(n) \\ &= -\frac{1}{2n^2} + \frac{2}{3n^3} - \frac{3}{4n^4} + \frac{4}{5n^5} - \frac{5}{6n^6} + \frac{6}{7n^7} - \frac{7}{8n^8} + \frac{8}{9} \frac{1}{n^9} \\ & \quad - \frac{9}{10} \frac{1}{n^{10}} + \frac{736265}{836136} \frac{1}{n^{11}} + O\left(\frac{1}{n^{12}}\right), \end{aligned}$$

which is the desired result. Substituting (2.8) and (2.13) into (2.7), we get

$$(2.14) \quad r_9(n) - r_9(n+1) = -\frac{262445}{9197496} \frac{1}{n^{11}} + O\left(\frac{1}{n^{12}}\right),$$

the rate of convergence of the  $(r_9(n) - \gamma)_{n \in \mathbb{N}}$  is  $n^{-10}$ , since

$$\lim_{n \rightarrow \infty} n^{10} (r_9(n) - \gamma) = -\frac{262445}{91974960}.$$

Next, we can use the **Step 1** to find  $a_{10}$ , and the **Step 2** to check  $a_{11}$  and  $a_{12}$ . It should be noted that Theorem 2 will provide their another proofs for  $a_{10}$  and  $a_{11}$ . So we omit the details here.

Finally, we check  $a_{13} = -\frac{306232774533}{179081182865}$ .

$$(2.15) \quad \begin{aligned} & R_{13}(n+1) - R_{13}(n) \\ &= -\frac{1}{2n^2} + \frac{2}{3n^3} - \frac{3}{4n^4} + \frac{4}{5n^5} - \frac{5}{6n^6} + \frac{6}{7n^7} - \frac{7}{8n^8} + \frac{8}{9} \frac{1}{n^9} \\ & \quad - \frac{9}{10} \frac{1}{n^{10}} + \frac{10}{11} \frac{1}{n^{11}} - \frac{11}{12} \frac{1}{n^{12}} + \frac{12}{13} \frac{1}{n^{13}} - \frac{13}{14} \frac{1}{n^{14}} \\ & \quad + \frac{1903648586623}{2576034146400} \frac{1}{n^{15}} + O\left(\frac{1}{n^{16}}\right). \end{aligned}$$

Substituting (2.8) and (2.15) into (2.7), one has

$$(2.16) \quad r_{13}(n) - r_{13}(n+1) = -\frac{500649950017}{2576034146400} \frac{1}{n^{15}} + O\left(\frac{1}{n^{16}}\right).$$

Since

$$\lim_{n \rightarrow \infty} n^{14} (r_{13}(n) - \gamma) = -\frac{71521421431}{5152068292800},$$

thus the rate of convergence of the  $(r_{13}(n) - \gamma)_{n \in \mathbb{N}}$  is  $n^{-14}$ .

This completes the proof of Theorem 1.  $\square$

*Remark 3.* In fact, if the assertion  $a_{13} = -\frac{306232774533}{179081182865}$  holds, then the other values  $a_j$  ( $1 \leq j \leq 12$ ) must be true. The following *Mathematica* program will generate  $R_{13}(n+1) - R_{13}(n)$  into power series in  $\frac{1}{n}$  with order 16:

Normal[Series[( $R_{13}[n+1] - R_{13}[n]$ )/.  $n \rightarrow 1/x$ , { $x, 0, 16$ }]]/.  $x \rightarrow 1/n$

*Remark 4.* It is a very interesting question to find  $a_k$  for  $k \geq 14$ . However, it seems impossible by the above method.

### 3 The Proof of Theorem 2

Before we prove the Theorem 2, let us give a simple inequality, which plays an important role of the proof.

**Lemma 2.** *Let  $f''(x)$  be a continuous function. If  $f''(x) > 0$ , then*

$$(3.1) \quad \int_a^{a+1} f(x) dx > f(a + 1/2).$$

*Proof.* Let  $x_0 = a + 1/2$ . By Taylor's formula, we have

$$\begin{aligned} \int_a^{a+1} f(x) dx &= \int_a^{a+1} \left( f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(\theta_x)(x - x_0)^2 \right) dx \\ &> \int_a^{a+1} (f(x_0) + f'(x_0)(x - x_0)) dx \\ &= f(a + 1/2). \end{aligned}$$

This completes the proof of Lemma 2.  $\square$

In the sequel, the notation  $P_k(x)$  means a polynomial of degree  $k$  in  $x$  with all of its non-zero coefficients positive, which may be different at each occurrence.

Let's begin to prove Theorem 2. Note  $r_{10}(\infty) = 0$ , it is easy to see

$$(3.2) \quad \gamma - r_{10}(n) = \sum_{m=n}^{\infty} (r_{10}(m+1) - r_{10}(m)) = \sum_{m=n}^{\infty} f(m),$$

where

$$f(m) = \frac{1}{m+1} - \ln\left(1 + \frac{1}{m}\right) - R_{10}(m+1) + R_{10}(m).$$

Let  $D_1 = \frac{2755095121}{6762022344}$ . By using the *Mathematica* software, we have

$$f'(x) + D_1 \frac{1}{(x+1)^{13}} = -\frac{P_{19}(x)(x-1) + 1619906998377 \cdots 5270931}{33810111720x(1+x)^{13}P_{10}^{(1)}(x)P_{10}^{(2)}(x)} < 0,$$

and

$$f'(x) + D_1 \frac{1}{(x+\frac{1}{2})^{13}} = \frac{P_{22}(x)}{4226263965x(1+x)^2(1+2x)^{13}P_{10}^{(3)}(x)P_{10}^{(4)}(x)} > 0.$$

Hence, we get the following inequalities for  $x \geq 1$ ,

$$(3.3) \quad D_1 \frac{1}{(x+1)^{13}} < -f'(x) < D_1 \frac{1}{(x+\frac{1}{2})^{13}}$$

Applying  $f(\infty) = 0$ , (3.3) and Lemma 2, we get

$$(3.4) \quad \begin{aligned} f(m) &= -\int_m^\infty f'(x)dx \leq D_1 \int_m^\infty \left(x + \frac{1}{2}\right)^{-13} dx \\ &= \frac{D_1}{12} \left(m + \frac{1}{2}\right)^{-12} \leq \frac{D_1}{12} \int_m^{m+1} x^{-12} dx. \end{aligned}$$

From (3.1) and (3.4) we obtain

$$(3.5) \quad \begin{aligned} \gamma - r_{10}(n) &\leq \sum_{m=n}^\infty \frac{D_1}{12} \int_m^{m+1} x^{-12} dx \\ &= \frac{D_1}{12} \int_n^\infty x^{-12} dx = \frac{D_1}{132} \frac{1}{n^{11}}. \end{aligned}$$

Similarly, we also have

$$\begin{aligned} f(m) &= -\int_m^\infty f'(x)dx \geq D_1 \int_m^\infty (x+1)^{-13} dx \\ &= \frac{D_1}{12} (m+1)^{-12} \geq \frac{D_1}{12} \int_{m+1}^{m+2} x^{-12} dx, \end{aligned}$$

and

$$(3.6) \quad \begin{aligned} \gamma - r_{10}(n) &\geq \sum_{m=n}^\infty \frac{D_1}{12} \int_{m+1}^{m+2} x^{-12} dx \\ &= \frac{D_1}{12} \int_{n+1}^\infty x^{-12} dx = \frac{D_1}{132} \frac{1}{(n+1)^{11}}. \end{aligned}$$



Combining (3.5) and (3.6) completes the proof of (1.6).

Note  $r_{11}(\infty) = 0$ , it is easy to deduce

$$(3.7) \quad r_{11}(n) - \gamma = \sum_{m=n}^{\infty} (r_{11}(m) - r_{11}(m+1)) = \sum_{m=n}^{\infty} g(m),$$

where

$$g(m) = \ln \left( 1 + \frac{1}{m} \right) - \frac{1}{m+1} - R_{11}(m) + R_{11}(m+1).$$

We write  $D_2 = \frac{20169451}{24495240}$ . By using the *Mathematica* software, we have

$$-g'(x) - D_2 \frac{1}{(x+1)^{14}} = \frac{P_{18}(x)}{24495240x^3(1+x)^{14}P_8^{(1)}(x)P_8^{(2)}(x)} > 0$$

and

$$-g'(x) - D_2 \frac{1}{(x+\frac{1}{2})^{14}} = -\frac{P_{19}(x)(x-1) + 4622005677839353997724676307741}{6123810x^3(1+x)^3(1+2x)^{14}P_8^{(3)}(x)P_8^{(4)}(x)} < 0.$$

Hence for  $x \geq 1$ ,

$$(3.8) \quad D_2 \frac{1}{(x+1)^{14}} < -g'(x) < D_2 \frac{1}{(x+\frac{1}{2})^{14}}.$$

Applying  $g(\infty) = 0$ , (3.8) and (3.1), we get

$$(3.9) \quad \begin{aligned} g(m) &= - \int_m^{\infty} g'(x) dx \leq D_2 \int_m^{\infty} \left( x + \frac{1}{2} \right)^{-14} dx \\ &= \frac{D_2}{13} \left( m + \frac{1}{2} \right)^{-13} \leq \frac{D_2}{13} \int_m^{m+1} x^{-13} dx. \end{aligned}$$

It follows from (3.7) and (3.9)

$$(3.10) \quad \begin{aligned} r_{11}(n) - \gamma &\leq \sum_{m=n}^{\infty} \frac{D_2}{13} \int_m^{m+1} x^{-13} dx \\ &= \frac{D_2}{13} \int_n^{\infty} x^{-13} dx = \frac{D_2}{156} \frac{1}{n^{12}}. \end{aligned}$$

Finally,

$$\begin{aligned} g(m) &= - \int_m^{\infty} g'(x) dx \geq D_2 \int_m^{\infty} (x+1)^{-14} dx \\ &= \frac{D_2}{13} (m+1)^{-13} \geq \frac{D_2}{13} \int_{m+1}^{m+2} x^{-13} dx. \end{aligned}$$

and

$$(3.11) \quad \begin{aligned} r_{11}(n) - \gamma &\geq \sum_{m=n}^{\infty} \frac{D_2}{13} \int_{m+1}^{m+2} x^{-13} dx \\ &= \frac{D_2}{13} \int_{n+1}^{\infty} x^{-13} dx = \frac{D_2}{156} \frac{1}{(n+1)^{12}}. \end{aligned}$$

Combining (3.10) and (3.11) completes the proof of (1.7).  $\square$

*Remark 5.* As an example, we give the *Mathematica Program* for the proof of the left-hand side of (3.3):

- (i) Together[D[f[x], {x, 1}]+D<sub>1</sub>(x+1)<sup>13</sup>];
- (ii) Take out the numerator P[x] of the above rational function, then manipulate the program: Apart[P[x]/(x-1)].

## 4 Competing Interests

The authors declare that they have no competing interests.

## 5 Authors' Contributions

Hongmin Xu conceived of the algorithm and helped to draft the manuscript. Xu You carried out the design of the program and drafted the manuscript. All authors read and approved the final manuscript.

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**Appendix** For the reader's convenience, we rewrite  $R_k(n)$  ( $k \leq 13$ ) with minimal

denominators as following.

$$\begin{aligned}
R_1(n) &= \frac{1}{2n}, \\
R_3(n) &= \frac{1}{2n} - \frac{1}{12} \frac{1}{n^2}, \\
R_5(n) &= \frac{1}{2n} - \frac{5}{6(1+10n^2)}, \\
R_7(n) &= \frac{1}{2n} - \frac{79}{1200} \frac{1}{n^2} - \frac{147}{400(10+21n^2)}, \\
R_9(n) &= \frac{1}{2n} - \frac{7(871+790n^2)}{20(241+3990n^2+3318n^4)}, \\
R_{11}(n) &= \frac{1}{2n} - \frac{52489}{894348} \frac{1}{n^2} - \frac{1237227621+584280400n^2}{4471740(3549+13020n^2+5302n^4)}, \\
R_{13}(n) &= \frac{1}{2n} - \frac{39577260671+66288226620n^2+15762446700n^4}{1260(20169451+434410620n^2+646328298n^4+150118540n^6)}.
\end{aligned}$$

$$\begin{aligned}
R_2(n) &= \frac{3}{6n+1}, \\
R_4(n) &= \frac{13+30n}{6(1+6n+10n^2)}, \\
R_6(n) &= \frac{5(281+348n+756n^2)}{6(79+600n+790n^2+1260n^3)}, \\
R_8(n) &= \frac{964337+2646000n+2599730n^2+2621220n^3}{20(19039+144600n+315210n^2+303660n^3+262122n^4)}, \\
R_{10}(n) &= \frac{7(108237701+208886046n+523341290n^2+210464400n^3+230000760n^4)}{20(12649849+107768934n+209431110n^2+395365320n^3+174158502n^4+161000532n^5)}, \\
R_{12}(n) &= (3604759235968501+11032319618513046n+17366281558290420n^2+19958033982902400n^3 \\
&\quad +7661417445218460n^4+4964130389017800n^5)/(1260(1058674313539+9019254081474n \\
&\quad +22801779033180n^2+33088387754520n^3+33925126033722n^4+13474242079452n^5 \\
&\quad +7879572046060n^6)).
\end{aligned}$$

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Hongmin Xu  
 Department of Mathematics and Physics,  
 Beijing Institute of Petro-Chemical Technology,  
 Beijing 102617, P. R. China  
 e-mail: xuhongmin@bipt.edu.cn

Xu You  
 1. School of Mathematics and System Science,  
 Beijing University of Aeronautics and Astronautics,  
 Beijing 100191, P. R. China  
 2. Department of Mathematics and Physics,

Beijing Institute of Petro-Chemical Technology,  
Beijing 102617, P. R. China  
e-mail: youxu@bipt.edu.cn